3.4 Real Zeros of Polynomials

① Rational Zeros

② Integer Zeros

③ Upper and Lower Bounds

Revised subtopics

① Zeros of a polynomial

② Descartes' Rule of Signs

① Rational Zeros

5. $P(x) = x^3 - 4x^2 + 3$

Rational Zeros Theorem: $\pm 3, \pm 1$

6. $Q(x) = x^4 - 3x^3 - 6x + 8$

Rational Zeros Theorem: $\pm 1, \pm 2, \pm 4, \pm 8$

7. $R(x) = 2x^5 + 3x^3 + 4x^2 - 8$

$\therefore \pm \frac{1}{2}, \pm 1, \pm 2, \pm 4, \pm 8$

② Integer Zeros

15. $P(x) = x^3 + 2x^2 - 13x + 10$

Rational Zero Theorem: $\pm 1, \pm 2, \pm 5, \pm 10$

1	1	2	-13	10
		2	4	-9
	1	4	-9	1

-1	1	2	-13	10
		-1	-1	14
	1	1	-14	24

2	1	2	-13	10
		2	8	-10
	1	4	-5	0

$p(x) = (x-2)(x^2 + 4x - 5)$

$= (x-2)(x+5)(x-1)$

$\therefore$ Real zeros are $1, 2, -5$

16. $P(x) = x^3 - 4x^2 - 19x - 14$

Rational Zeros Theorem: $\pm 1, \pm 2, \pm 7, \pm 14$

$$\begin{array}{r|rrrr} 1 & 1 & -4 & -19 & -14 \\ & & 1 & -3 & \\ \hline & 1 & -3 & -22 & \end{array}$$

$$\begin{array}{r|rrrr} 2 & 1 & -4 & -19 & -14 \\ & & 2 & -4 & \\ \hline & 1 & -2 & -23 & \end{array}$$

$$\begin{array}{r|rrrr} 7 & 1 & -4 & -19 & -14 \\ & & 7 & 21 & 14 \\ \hline & 1 & 3 & 2 & 0 \end{array}$$

$$P(x) = (x-7)(x^2+3x+2)$$
$$= (x-7)(x+2)(x+1)$$

17. $P(x) = x^3 + 3x^2 - 4$

Rational Zeros Theorem: $\pm 1, \pm 2, \pm 4$

1	1	3	0	−4
		1	4	4
	1	4	4	0

$\therefore p(x) = (x-1)(x^2+4x+4)$

$= (x-1)(x+2)^2$

Zeros: $-2, 1$

18. $p(x) = x^3 - 3x - 2$

Rational Zeros Theorem: $\pm 1, \pm 2$

2	1	0	−3	−2
		2	4	2
	1	2	1	0

$\therefore p(x) = (x-2)(x^2+2x+1)$

$= (x-2)(x+1)^2$

Zeros: $-1, 2$

19. $P(x) = x^3 - 6x^2 + 12x - 8$

Rational Zeros Theorem: $\pm 1, \pm 2, \pm 4, \pm 8$

DRS: 3 or 1 positive zero, no negative zero

$$\begin{array}{r|rrrr} 2 & 1 & -6 & 12 & -8 \\ & & 2 & -8 & 8 \\ \hline & 1 & -4 & 4 & 0 \end{array}$$

$$P(x) = (x-2)(x^2 - 4x + 4)$$
$$= (x-2)(x-2)^2$$
$$= (x-2)^3$$

$\therefore$ 2

29. $P(x) = 4x^4 - 37x^2 + 9$

Rational Zero Theorem: $\pm 1, \pm 3, \pm 9, \pm \frac{1}{2}, \pm \frac{1}{4}, \pm \frac{3}{2}, \pm \frac{3}{4}, \pm \frac{9}{2}, \pm \frac{9}{4}$

Descartes Rule of Sign: 2 or 0 positive rational zero

2 or 0 negative rational zero

3	4	0	-37	0	9
		12	36	-3	-9
	4	12	-1	-3	0

$P(x) = (x-3)(4x^3 + 12x^2 - x - 3)$

1/2	4	12	-1	-3
		2	7	3
	4	14	6	0

$$p(x) = (x-3)(2x-1)(4x^2+14x+6)$$
$$= (x-3)(2x-1)(4x+2)(x+3)$$

$$\therefore x = -3, -\frac{1}{2}, \frac{1}{2}, 3$$

Descartes' Rule of Signs

63. $P(x) = x^3 - x^2 - x - 3$

1 variation in sign, ∴ 1 positive real zero

$$P(-x) = (-x)^3 - (-x)^2 - (-x) - 3$$
$$= -x^3 - x^2 + x - 3$$

2 variations in sign, ∴ 2 or 0 negative real zeros

∴ 1 or 3 total number of real zeros possible

③ Upper and Lower Bounds

69. $P(x) = 2x^3 + 5x^2 + x - 2$; $a = -3, b = 1$

Test for Lower Bound

-3	2	5	1	-2
		-6	3	-12
	2	-1	4	-14

∵ alternating signs
∴ -3 is a lower bound

Test for Upper Bound

1	2	5	1	-2
		2	7	8
	2	7	8	6

∵ non-negative entries,
∴ 1 is an upper bound

70. $P(x) = x^4 - 2x^3 - 9x^2 + 2x + 8$; $a = -3$, $b = 5$

5	1	-2	-9	2	8
		5	15	30	160
	1	3	6	32	168

∴ non-negative entries, upper bound

-3	1	-2	-9	2	8
		-3	15	-18	48
	1	-5	6	-16	56

∴ alternating signs, lower bound

71. $P(x) = 8x^3 + 10x^2 - 39x + 9$; $a = -3, b = 2$

-3	8	10	-39	9
		-24	42	-9
	8	-14	3	0

∴ lower bound,

-3 is also a zero

2	8	10	-39	9
		16	52	26
	8	26	13	35

∴ upper bound

72. $P(x) = 3x^4 - 17x^3 + 24x^2 - 9x + 1$; $a = 0, b = 6$

0	3	-17	24	-9	1
		0	0	0	0
	3	-17	24	-9	1

∴ Alternate signs, lower bound

6	3	-17	24	-9	1
		18	6	180	
	3	1	30	171	

∴ all non-negative entries, upper bound

Descartes' Rules of Signs

63. $P(x) = x^3 - x^2 - x - 3$

$\therefore$ 1 positive real zero

$P(-x) = (-x)^3 - (-x)^2 - (-x) - 3$

$= -x^3 - x^2 + x - 3$

$\therefore$ 2 or 0 negative real zeros

$\therefore$ 1 or 3 possible total real zeros

Zeros of a Polynomial 81 - 86

81. $P(x) = 2x^4 + 3x^3 - 4x^2 - 3x + 2$

Rational Zero Theorem: $\pm 1, \pm 2, \pm \frac{1}{2}$

1	2	3	-4	-3	2
		2	5	1	-2
	2	5	1	-2	0

$P(x) = (x-1)(2x^3 + 5x^2 + x - 2)$

-1	2	5	1	-2
		-2	-3	2
	2	3	-2	0

$P(x) = (x-1)(x+1)(2x^2 + 3x - 2)$

$= (x-1)(x+1)(2x-1)(x+2)$

3.5 Complex Zeros and the Fundamental Theorem of Algebra

① Complete Factorisation

② Finding Complex Zeros

② Finding Complex Zeros

31. $Q(x) = x^4 + 2x^2 + 1$

$= (x^2+1)^2$

$= ((x+i)(x-i))^2$

$= (x+i)^2(x-i)^2$

$x^2+1=0$

$x^2=-1$

$x=\pm i$

Zeros: $-i, i$
each of multiplicity 2

37. Zeros: $1+i, 1-i$

Complete Factorisation Theorem:

$p(x) = a(x-c_1)(x-c_2)\cdots(x-c_n)$

$p(x) = a(x-(1+i))(x-(1-i))$

$= a(x-1-i)(x-1+i)$

$= a((x-1)^2 - i^2)$

$= a((x-1)^2+1)$

$= a(x^2-2x+1+1)$

$= a(x^2-2x+2)$

Let $a=1$, $\therefore p(x) = x^2-2x+2$

Complex Zeros Come in Conjugate Pairs

41. Zeros : $2, i$

Conjugate Zeros Theorem

Zeros : $2, i, -i$

$$P(x) = (x-2)(x-i)(x+i)$$
$$= (x-2)(x^2-i^2)$$
$$= (x-2)(x^2+1)$$
$$= x^3+x-2x^2-2$$
$$\therefore P(x) = x^3-2x^2+x-2$$

Linear and Quadratic Factors

67. $P(x) = x^4+8x^2-9$

(a) $P(x) = (x^2-1)(x^2+9)$
$$= (x+1)(x-1)(x^2+9)$$

(b) $P(x) = (x+1)(x-1)(x+\sqrt{3}i)(x-\sqrt{3}i)$

$x^2+9=0$
$x^2=-3$
$x=\pm\sqrt{3}i$

① Complete Factorisation

7. $P(x) = x^4 + 4x^2$

$x^2 = 0$
$x = 0$

(a) $p(x) = x^4 + 4x^2$
$= x^2(x^2 + 4)$

$x^2 + 4 = 0$
$x^2 = -4$
$x = \pm\sqrt{-4}$
$= \pm\sqrt{4i^2}$
$= \pm 2i$

$\therefore 0, -2i, 2i$

(b) $p(x) = x^2(x + 2i)(x - 2i)$

8. $P(x) = x^5 + 9x^3$
$= x^3(x^2 + 9)$
$= x^3(x - 3i)(x + 3i)$

$x^2 + 9 = 0$
$x^2 = -9$
$x = \pm 3i$

$\therefore 0, \pm 3i$

Multiplicity: 3, 1, 1

9. $P(x) = x^3 - 2x^2 + 2x$

Rational Zeros Theorem: $\pm 1, \pm 2$

1	1	−2	2
		1	−1
	1	−1	1

2	1	−2	2
		2	0
	1	0	2

∴ upper bound

−1	1	−2	2
		−1	3
	1	−3	5

∴ Lower bound

(b)

$P(x) = x(x^2 - 2x + 2)$

$= x(x-1-i)(x-1+i)$

$x^2 - 2x + 2 = 0$

$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \dfrac{2 \pm \sqrt{4-8}}{2}$

$= \dfrac{2 \pm \sqrt{4i^2}}{2}$

$= 1 \pm i$

(a)

∴ Zeros: $0, 1+i, 1-i$

Multiplicity: 1, 1, 1

10. $P(x) = x^3 + x^2 + x$

(a) $P(x) = x(x^2 + x + 1)$

$x^2 + x + 1 = 0$

$x = \frac{-1 \pm \sqrt{1-4}}{2}$

$= \frac{-1 \pm \sqrt{-3}}{2}$

$= \frac{-1 \pm \sqrt{3}i}{2}$

Zeros: $x = 0, \frac{-1+\sqrt{3}i}{2}, \frac{-1-\sqrt{3}i}{2}$

(b)

$P(x) = x\left(x + \frac{1-\sqrt{3}i}{2}\right)\left(x + \frac{1+\sqrt{3}i}{2}\right)$

② Finding Complex Zeros

47. $P(x) = x^3 + 2x^2 + 4x + 8$

Rational Zero Theorem: $\pm 1, \pm 2, \pm 4, \pm 8$

1	1	2	4	8
		1	3	7
	1	3	7	15

∵ non-negative entries
∴ 1 is an upper bound

−1	1	2	4	8
		−1	−1	−3
	1	1	3	5

−2	1	2	4	8
		−2	0	−8
	1	0	4	0

$x^2 + 4 = 0$

$P(x) = (x+2)(x^2+4)$ $\quad x^2 = -4$

$= (x+2)(x+2i)(x-2i)$ $\quad x = \pm 2i$

48. $P(x) = x^3 - 7x^2 + 17x - 15$

Rational Zero Theorem: $\pm 1, \pm 3, \pm 5, \pm 15$

1	1	-7	17	-15
		1	-6	11
	1	-6	11	-4

3	1	-7	17	-15
		3	-12	15
	1	-4	5	0

$P(x) = (x-3)(x^2 - 4x + 5)$

$= (x-3)(x-(2+2i))(x-(2-2i))$

$= (x-3)(x-2-2i)(x-2+2i)$

$x^2 - 4x + 5 = 0$

$x = \frac{4 \pm \sqrt{11 - 20}}{2}$

$= \frac{4 \pm \sqrt{4i^2}}{2}$

$= 2 \pm 2i$

$\therefore 3, 2+2i, 2-2i$

49. $P(x) = x^3 - 2x^2 + 2x - 1$

Rational Zeros Theorem : ± 1

1	1	-2	2	-1
		1	-1	1
	1	-1	1	0

$P(x) = (x-1)(x^2 - x + 1)$

$x^2 - x + 1 = 0$

$x = \dfrac{1 \pm \sqrt{1-4}}{2}$

$= \dfrac{1 \pm \sqrt{3}\,i}{2}$

$\therefore 1, \ \dfrac{1+\sqrt{3}\,i}{2}, \ \dfrac{1-\sqrt{3}\,i}{2}$

50. $P(x) = x^3 + 7x^2 + 18x + 18$

Rational Zeros Theorem ; $\pm 1, \pm 2, \pm 3, \pm 6, \pm 9, \pm 18$

2	1	7	18	18
		2	18	72
	1	9	36	90

$\therefore$ upper bound

1	1	7	18	18
		1	8	26
	1	8	26	44

$$
\begin{array}{r|rrrr}
-3 & 1 & 7 & 18 & 18 \\
 & & -3 & -12 & -18 \\
\hline
 & 1 & 4 & 6 & 0
\end{array}
$$

$x^2+4x+6=0$

$x = \dfrac{-4 \pm \sqrt{16-24}}{2}$

$= \dfrac{-4 \pm \sqrt{8i^2}}{2}$

$= -2 \pm \sqrt{2}i$

$\therefore P(x) = (x-(-3))(x^2+4x+6)$

$= (x+3)(x+2-\sqrt{2}i)$

$(x+2+\sqrt{2}i)$

51. $P(x) = x^3 - 3x^2 + 3x - 2$

Rational Zeros Theorem: $\pm 1, \pm 2$

$$
\begin{array}{r|rrrr}
1 & 1 & -3 & 3 & -2 \\
 & & 1 & -2 & 1 \\
\hline
 & 1 & -2 & 1 & -1
\end{array}
$$

$$
\begin{array}{r|rrrr}
2 & 1 & -3 & 3 & -2 \\
 & & 2 & -2 & 2 \\
\hline
 & 1 & -1 & 1 & 0
\end{array}
$$

$x^2 - x + 1 = 0$

$x = \dfrac{1 \pm \sqrt{1-4}}{2}$

$= \dfrac{1 \pm \sqrt{3}i}{2}$

$P(x) = (x-2)(x^2-x+1)$

$\therefore \ 2, \ \dfrac{1+\sqrt{3}i}{2}, \ \dfrac{1-\sqrt{3}i}{2}$

52. $P(x) = x^3 - x - 6$

Descartes' Rule of Signs: 1 positive real zero

2 or 0 negative real zero

Rational Zeros Theorem: $\pm 1, \pm 2, \pm 3, \pm 6$

$$\begin{array}{r|rrrr} 1 & 1 & 0 & -1 & -6 \\ & & 1 & 1 & 0 \\ \hline & 1 & 1 & 0 & -6 \end{array}$$

$$\begin{array}{r|rrrr} 2 & 1 & 0 & -1 & -6 \\ & & 2 & 4 & 6 \\ \hline & 1 & 2 & 3 & 0 \end{array}$$

$x^2 + 2x + 3 = 0$

$x = \frac{-2 \pm \sqrt{4-12}}{2}$

$= \frac{-2 \pm 2\sqrt{2}i}{2}$

$= -1 \pm \sqrt{2}i$

$P(x) = (x-2)(x^2+2x+3)$

$= (x-2)(x+1-\sqrt{2}i)(x+1+\sqrt{2}i)$

$\therefore$ Zeros: $2, -1+\sqrt{2}i, -1-\sqrt{2}i$

53. $P(x) = 2x^3 + 7x^2 + 12x + 9$

RZT: $\pm 1, \pm 3, \pm 9, \pm \frac{1}{2}, \pm \frac{3}{2}, \pm \frac{9}{2}$

DRS: No positive real zero, 3 or 1 negative real zero

$$\begin{array}{r|rrrr} -3 & 2 & 7 & 12 & 9 \\ & & -6 & -3 & -27 \\ \hline & 2 & 1 & 9 & -18 \end{array}$$

$$\begin{array}{r|rrrr} -9 & 2 & 7 & 12 & 9 \\ & & -18 & 99 & -999 \\ \hline & 2 & -11 & 111 & -981 \end{array}$$

Lower bound

$$\begin{array}{r|rrrr} -\frac{9}{2} & 2 & 7 & 12 & 9 \\ & & -9 & 9 & \\ \hline & 2 & -2 & 21 & \end{array}$$

$$\begin{array}{r|rrrr} -\frac{3}{2} & 2 & 7 & 12 & 9 \\ & & -3 & -6 & -9 \\ \hline & 2 & 4 & 6 & 0 \end{array}$$

$$P(x) = (2x-3)(2x^2+4x+6)$$

$$= (2x-3)(x+1-\sqrt{2}i)(x+1+\sqrt{2}i)$$

$$\therefore\ -\frac{3}{2},\ -1+\sqrt{2}i,\ -1-\sqrt{2}i$$

$$4x^2+8x+12=0$$

$$x^2+2x+3=0$$

$$x = \frac{-2 \pm \sqrt{4-12}}{2}$$

$$= \frac{-2 \pm 2\sqrt{2}i}{2}$$

$$= -1 \pm \sqrt{2}i$$

3.6 Rational Functions

① Graphing Rational Functions

Table of Values

9. $r(x) = \dfrac{x}{x-2}$

(a)

x	$r(x)$
1.5	-3
1.9	-19
1.99	-199
1.999	-1999

x	$r(x)$
2.5	5
2.1	21
2.01	201
2.001	2001

$r(1.5) = \dfrac{1.5}{1.5-2}$
$= -3$

$r(1.9) = \dfrac{1.9}{1.9-2}$
$= -19$

$r(1.99) = \dfrac{1.99}{1.99-2}$
$= -199$

$r(1.999) = -1999$

$r(2.5) = \dfrac{2.5}{2.5-2}$
$= 5$

$r(2.1) = \dfrac{2.1}{2.1-2}$
$= 21$

$r(2.01) = \dfrac{2.01}{2.01-2}$
$= 201$

$r(2.001) = \dfrac{2.001}{2.001-2}$
$= 2001$

(b) $r(x) \to -\infty$, as $x \to 2^-$

$r(x) \to \infty$, as $x \to 2^+$

(c)

Table 3

x	$r(x)$
10	2
50	1.042
100	1.020
1000	1.002

Table 4

x	$r(x)$
-10	0.833
-50	0.962
-100	0.980
-1000	0.998

$$r(10) = \frac{10}{10-2} = 2$$

$$r(50) = \frac{50}{50-2} = 1.042$$

$$r(100) = \frac{100}{100-2} = 1.020$$

$$r(1000) = \frac{1000}{1000-2} = 1.002$$

$$r(-10) = \frac{-10}{-10-2} = 0.833$$

$$r(-50) = \frac{-50}{-50-2} = 0.962$$

$$r(-100) = \frac{-100}{-100-2} = 0.980$$

$$r(-1000) = \frac{-1000}{-1000-2} = 0.998$$

$r(x) \to 1$, as $x \to \infty$

$r(x) \to 1$, as $x \to -\infty$

Graphing Rational Functions Using Transformations

15. $s(x) = \dfrac{3}{x+1}$

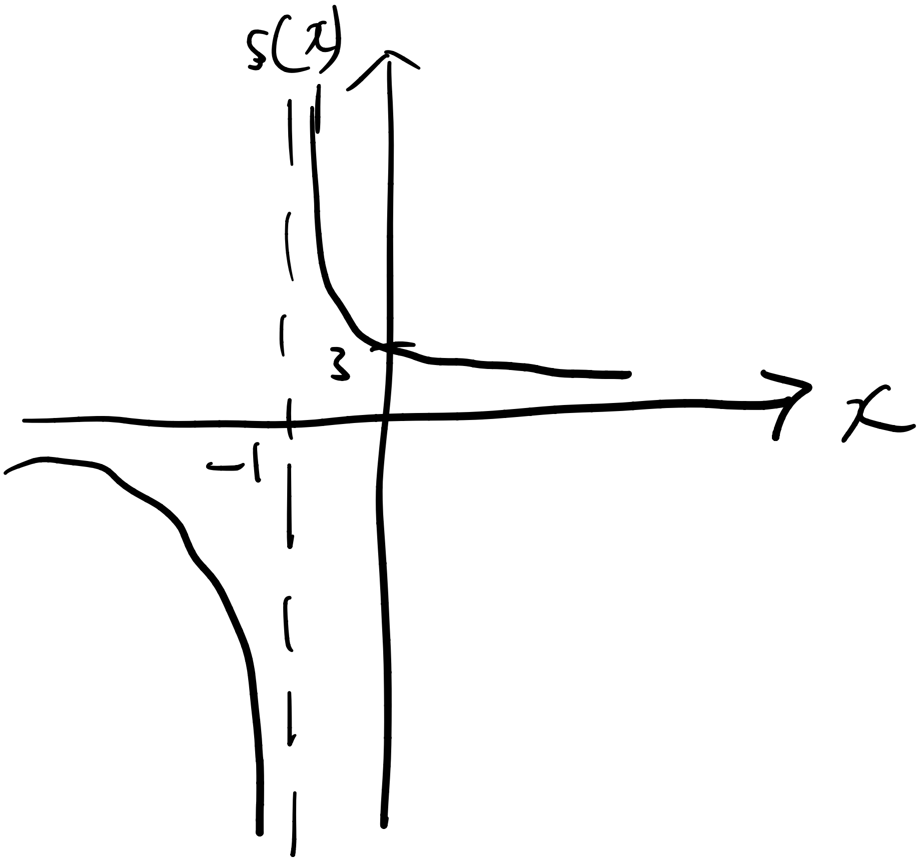
Domain: $\{x \mid x \neq -1\}$

Range: $\{y \mid y \neq 0\}$

$y = \dfrac{1}{x}$

$f(x) = \dfrac{1}{x}$

$$s(x) = \frac{3}{x+1}$$
$$= 3\left(\frac{1}{x+1}\right)$$
$$= 3f(x+1)$$



17. $t(x) = \dfrac{2x-3}{x-2}$

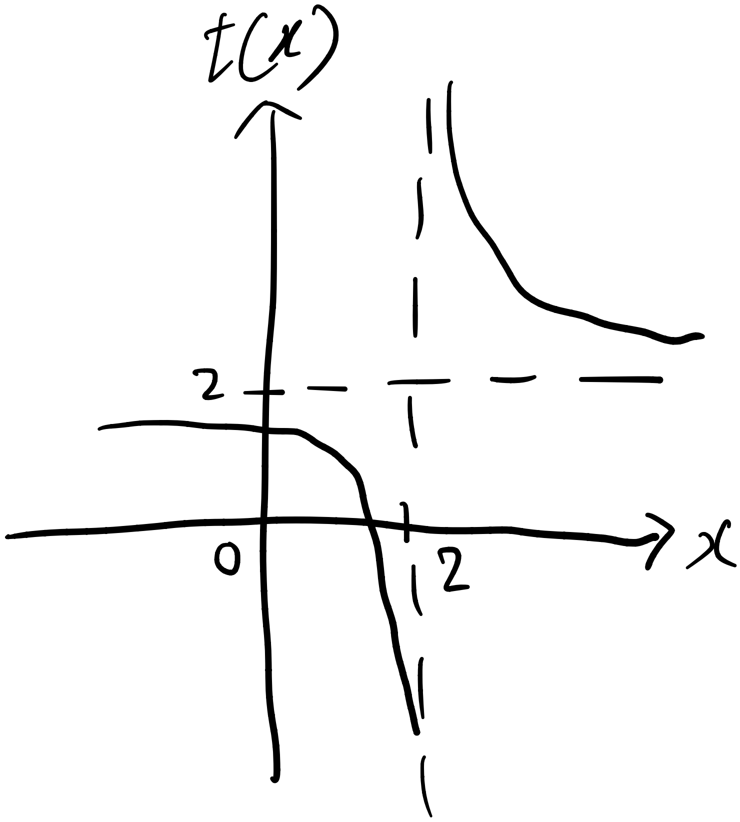
$y = \dfrac{1}{x}$

$f(x) = \dfrac{1}{x}$

2	2	−3
		4
	2	1

$t(x) = 2 + \dfrac{1}{x-2}$

$t(x) = 2 + f(x-2)$



Graphing Rational Functions

45. $r(x) = \dfrac{3x^2 - 12x + 13}{x^2 - 4x + 4}$ $\qquad r(x) = \dfrac{3x^2 - 12x + 13}{(x-2)^2}$

$= 3 + \dfrac{1}{x^2 - 4x + 4}$

$= 3 + \dfrac{1}{(x-2)^2}$

if $f(x) = \dfrac{1}{x}$

$= 3 + f((x-2)^2)$

$x \to 2^-$

x	$r(x)$
1	4
1.4	5.78
1.8	28
1.9	103

$x \to 2^+$

x	$r(x)$
3	4
2.4	9.25
2.1	103
2.01	10003

$r(2.1) = 103$

$r(2.01) = 10003$

$r(1) = \dfrac{3 - 12 + 13}{1} = 4$

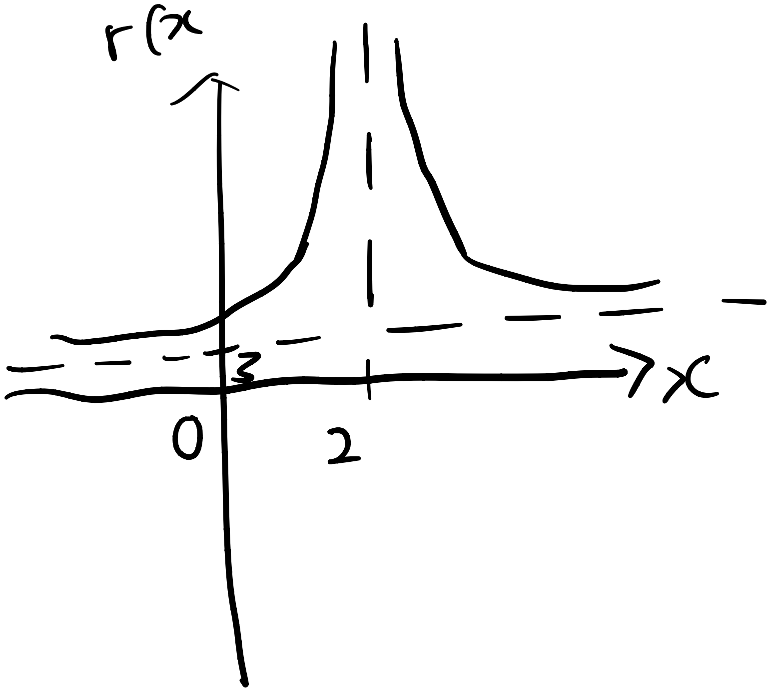
$r(1.4) = 5.78$

$r(1.8) = 28$

$r(1.9) = 103$

$r(3) = \dfrac{27 - 36 + 13}{1} = 4$

$r(2.4) = 9.25$



$$r(x) = \frac{3 - \frac{12}{x} + \frac{13}{x^2}}{1 - \frac{4}{x} + \frac{4}{x^2}}$$

As $x \to \infty$

$$r(x) = \frac{3}{1} = 3$$

33. $$r(x) = \frac{3x+1}{4x^2+1}$$

$$= \frac{3 + \frac{1}{x}}{4x + \frac{1}{x}}$$

$4x^2 + 1 = 0$

$4x^2 = -1$

$x^2 = -\frac{1}{4}$

$x = \pm \frac{1}{2} i$

As $x \to \infty$, $r(x) \to 0$

As $x \to -\infty$, $r(x) \to 0$

$\therefore$ Horizontal asymptote : 0,

No vertical asymptote since $4x^2+1 > 0$ for all x

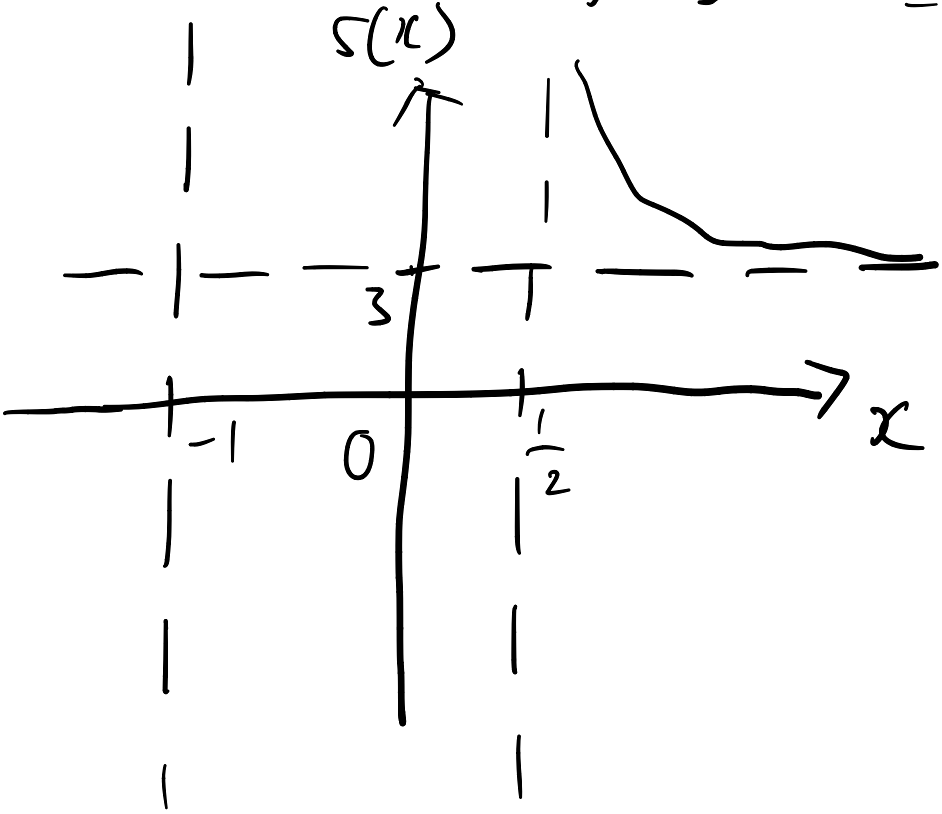
35. $s(x) = \dfrac{6x^2+1}{2x^2+x-1}$

$= \dfrac{6x^2+1}{(2x-1)(x+1)}$

$s(x) = \dfrac{6+\frac{1}{x^2}}{2+\frac{1}{x}-\frac{1}{x^2}}$

Vertical asymptote: $x = \frac{1}{2}, -1$

Horizontal asymptote: as $x \to \infty$, $s(x) = 3$



$s\left(\frac{3}{4}\right)$

$= \dfrac{6\left(\frac{9}{16}\right)+1}{2\left(\frac{9}{16}\right)+\left(\frac{3}{4}\right)-1}$

$= \dfrac{\frac{27}{8}+\frac{8}{8}}{\frac{7}{8}}$

$= 5$

$s\left(\frac{5}{8}\right)$

$= \dfrac{6\left(\frac{25}{64}\right)+\frac{32}{32}}{2\left(\frac{25}{64}\right)+\left(\frac{25}{64}\right)-\frac{64}{64}}$

$= \dfrac{\frac{107}{32}}{\frac{36}{64}}$

$= \dfrac{214}{36}$

$= \dfrac{107}{18}$

$s\left(\frac{3}{8}\right) =$

53. $r(x) = \frac{(x-1)(x+2)}{(x+1)(x-3)}$

Vertical Asymptote : $x = -1, 3$

$r(x) = \frac{x^2+x-2}{x^2-2x-3}$

Horizontal Asymptote : $y = 1$

x-intercepts : $x = -2, 1$

$x \to -1^+$

when $x = -0.9$,

$r(-0.9) = \frac{(-)(+)}{(+)(-)} = +$

$x \to -1^-$

$r(-1.1) = \frac{(-)(+)}{(-)(-)} = -$

$x \to 3^+$

$r(3.1) = \frac{(+)(+)}{(+)(+)}$

$= +$

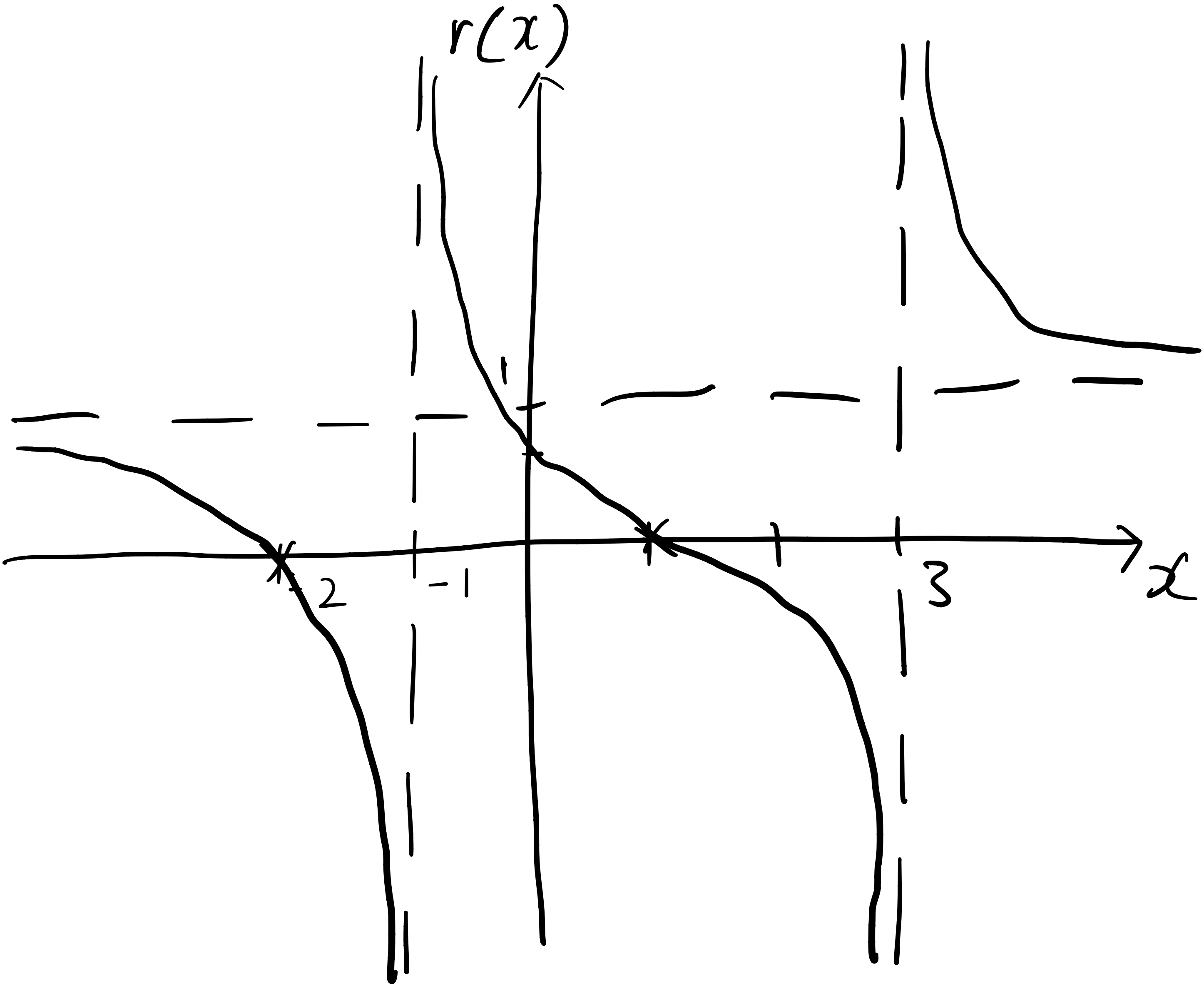
$x \to 3^-$

$r(2.9) = \frac{(+)(+)}{(+)(-)}$

$= -$

y-intercept : $r(0) = \frac{(-1)(2)}{(1)(-3)}$

$= \frac{2}{3}$



Domain : $\{x \mid x \neq -1, 3\}$

Range : $\{y \mid y \in R\}$

59. $r(x) = \dfrac{x^2 - 2x + 1}{x^2 + 2x + 1}$

$= \dfrac{(x-1)^2}{(x+1)^2}$

Vertical Asymptote: $x = -1$

Horizontal Asymptote: $y = \frac{1}{1} = 1$

y-interrcept: $r(0) = \dfrac{0 - 0 + 1}{0 + 0 + 1} = 1$

x-intercept: $x - 1 = 0$

$x = 1$

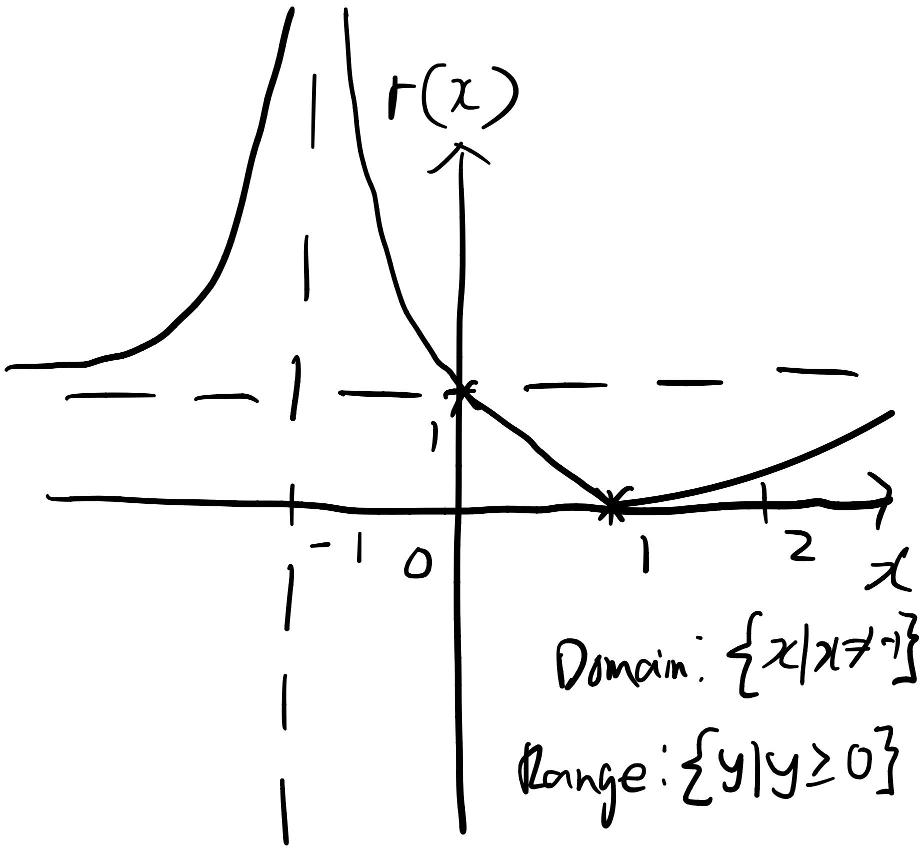
$x \to -1^-$

$r(-1.1) = +$

$x \to -1^+$

$r(-0.9) = +$

x	$r(x)$
2	1/9
3	1/4
4	9/25
1.2	0.008



Domain: $\{x \mid x \neq -1\}$

Range: $\{y \mid y \geq 0\}$

Common Factors in Numerator and Denominator

63. $r(x) = \dfrac{x^2+4x-5}{x^2+x-2}$

$= \dfrac{(x+5)(x-1)}{(x+2)(x-1)}$

$= \dfrac{x+5}{x+2} \quad x \neq 1$

Vertical asymptotes: $x = -2$

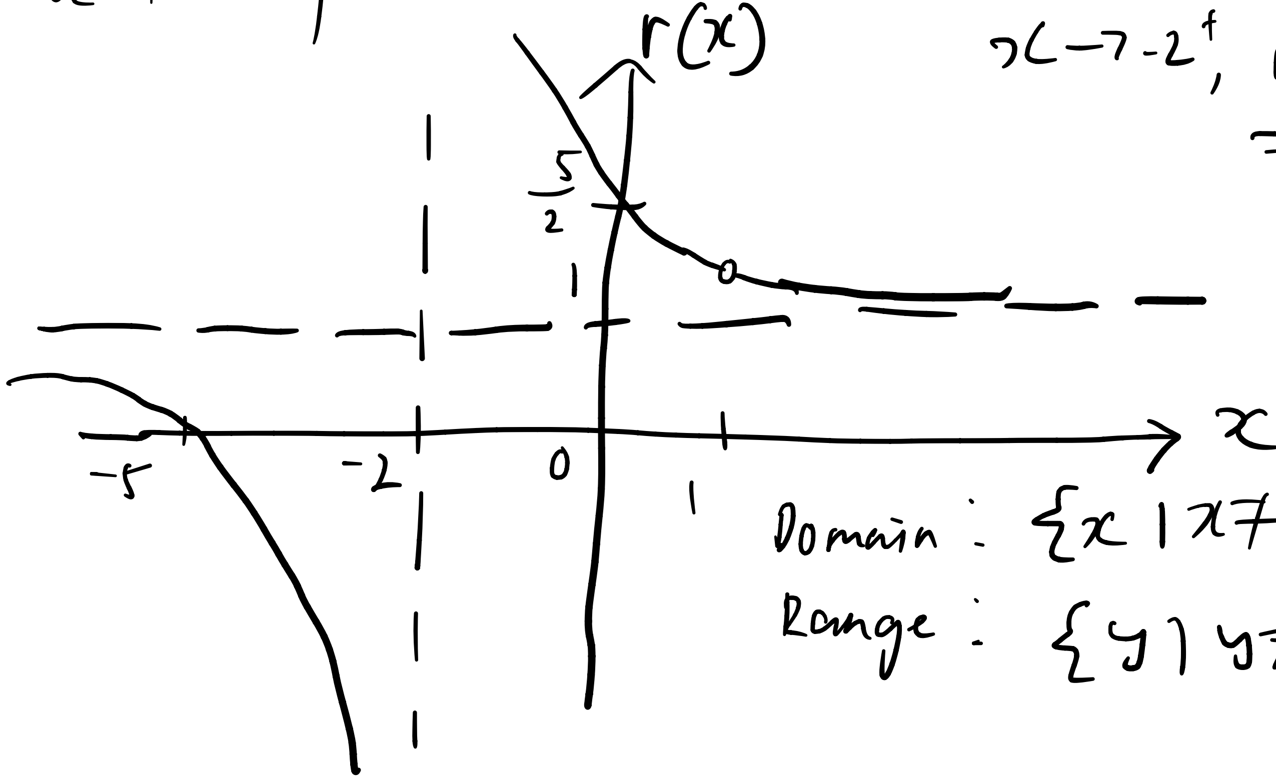
Horizontal asymtotes: $y = \frac{1}{1} = 1$

y-interrept: $y = \frac{5}{2}$

x-interrept: $x = -5$

$x \to -2^-$, $r(-2.1) = \dfrac{2.9}{-0.1}$

$x \to -2^+$, $r(-1.9) = \dfrac{3.1}{0.1}$



Domain: $\{x \mid x \neq -2, 1\}$

Range: $\{y \mid y \neq 1, 2\}$

(i) Graphing Rational Functions

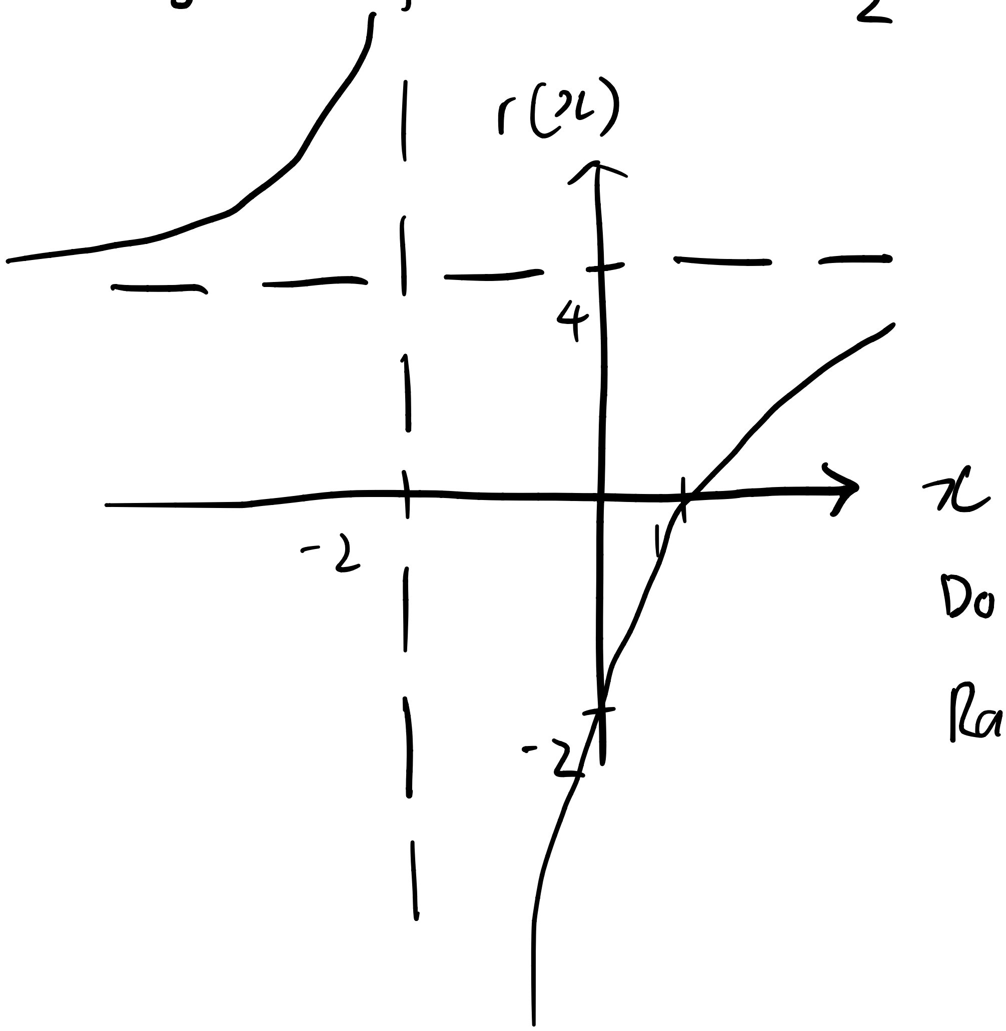
43. $r(x) = \dfrac{4x - 4}{x + 2}$

Vertical asymptote: $x = -2$

Horizontal asymptote: $y = \frac{4}{1} = 4$

x-intercept: $x = 1$

y-intercept: $y = \frac{-4}{2} = -2$



$x \to -2^{+}, y \to -\infty$

$x \to -2^{-}, y \to \infty$

Domain: $\{x \mid x \neq -2\}$

Range: $\{y \mid y \neq 4\}$

44. $r(x) = \dfrac{2x+6}{-6x+3}$

vertical asymptote: $x = \frac{1}{2}$

Horizontal asymptote: $y = -\frac{1}{3}$

$x \to \frac{1}{2}^-,\ y \to \infty$

$x \to \frac{1}{2}^+,\ y \to -\infty$

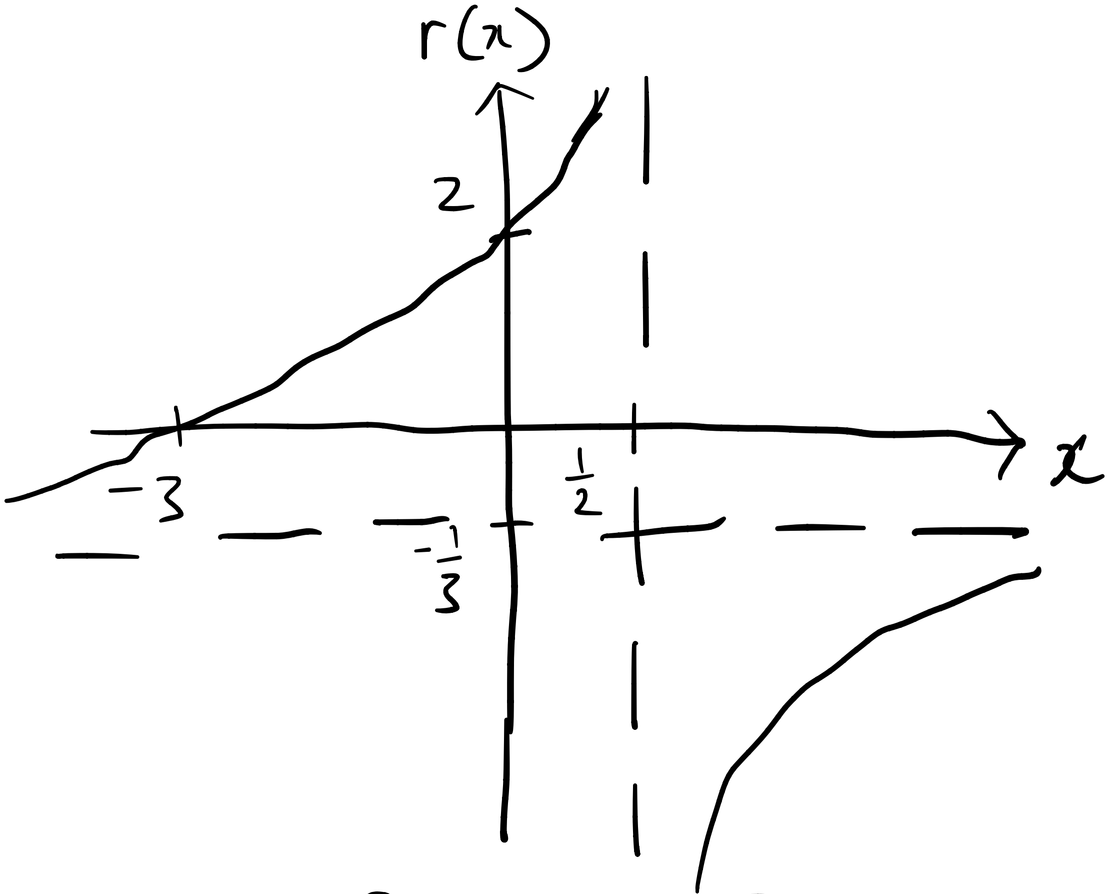
$x = \frac{3}{4},\ y = \dfrac{\frac{6}{4} + \frac{24}{4}}{-\frac{9}{2} + \frac{6}{2}}$

$= \frac{15}{2} \div -\frac{3}{2}$

$= -5$

x-intercept: $x = -3$

y-intercept: $y = 2$



Domain: $\{x \mid x \neq \frac{1}{2}\}$

Range: $\{y \mid y \neq -\frac{1}{3}\}$

45. $r(x) = \dfrac{3x^2 - 12x + 13}{x^2 - 4x + 4}$

$= \dfrac{3x^2 - 12x + 13}{(x-2)^2}$

$x \to 2^-, y \to \infty$

$x \to 2^+, y \to \infty$

Vertical Asymptote: $x = 2$

Horizontal Asymptote: $y = \frac{3}{1} = 3$

x-intercept: $3x^2 - 12x + 13 = 0$

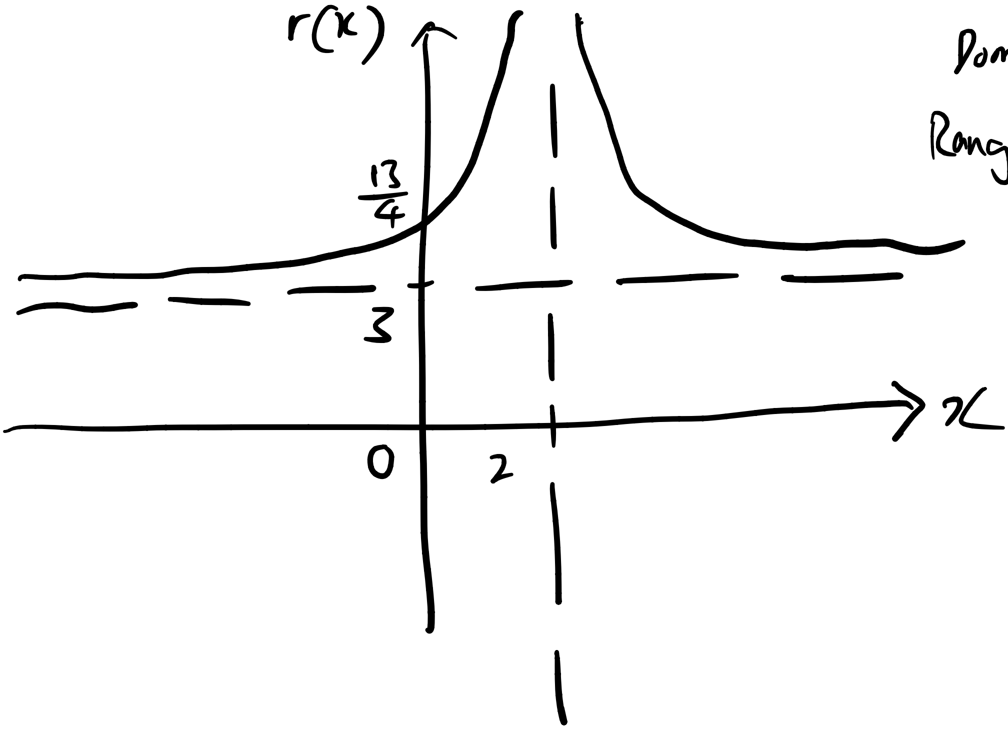
$x = \dfrac{12 \pm \sqrt{144 - 156}}{6}$

$= \dfrac{12 \pm \sqrt{12 i^2}}{6}$

$= \dfrac{12 \pm 2\sqrt{3} i}{6}$

$= \dfrac{6 \pm \sqrt{3} i}{3}$

y-intercept: $r(0) = \dfrac{13}{4}$



Domain : $\{x \mid x \neq 2\}$

Range : $\{y \mid y > 3\}$

46. $r(x) = \dfrac{-2x^2 - 8x - 9}{x^2 + 4x + 4}$

$= \dfrac{-2x^2 - 8x - 9}{(x+2)^2}$

As $x \to -2^-$, $y \to -\infty$

As $x \to -2^+$, $y \to \infty$

Vertical asymptote : $x = -2$

Horizontal asymptote : $y = \frac{-2}{1} = -2$

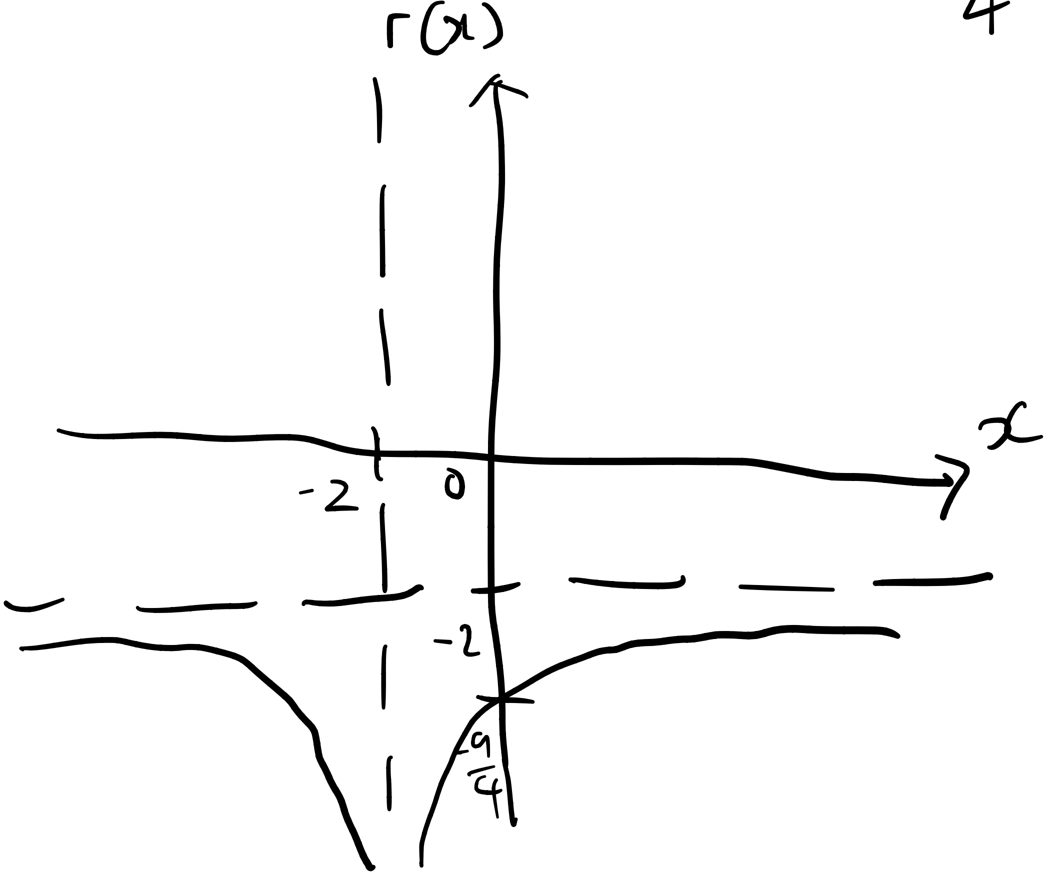
x-intercept : $-2x^2 - 8x - 9 = 0$

$x = \dfrac{8 \pm \sqrt{64 - 72}}{-4}$

$= \dfrac{-8 \pm 2\sqrt{2}i}{4}$

$= \dfrac{-4 \pm \sqrt{2}i}{2}$

y-intercept: $r(0) = \frac{-9}{4} = -\frac{9}{4}$



Domain: $\{x \mid x \neq -2\}$

Range: $\{y \mid y < -2\}$

47. $r(x) = \frac{-x^2 + 8x - 18}{x^2 - 8x + 16}$

$= \frac{-x^2 + 8x - 18}{(x-4)^2}$

Vertical asymptote: $x = 4$

y-asymptote: $r(x) = \frac{-1 + \frac{8}{x} - \frac{18}{x^2}}{1 - \frac{8}{x} + \frac{16}{x^2}}$

as $x \to \infty$, $r(x) \to \frac{-1}{1} = -1$

as $x \to 4^-$, $r(x) \to \infty$

$x = 3.9$, $r(x) = 3399$

as $x \to 4^+$, $r(x) \to \infty$

$x = 4.1$, $r(x) = 3599$

$-x^2 + 8x - 18 = 0$

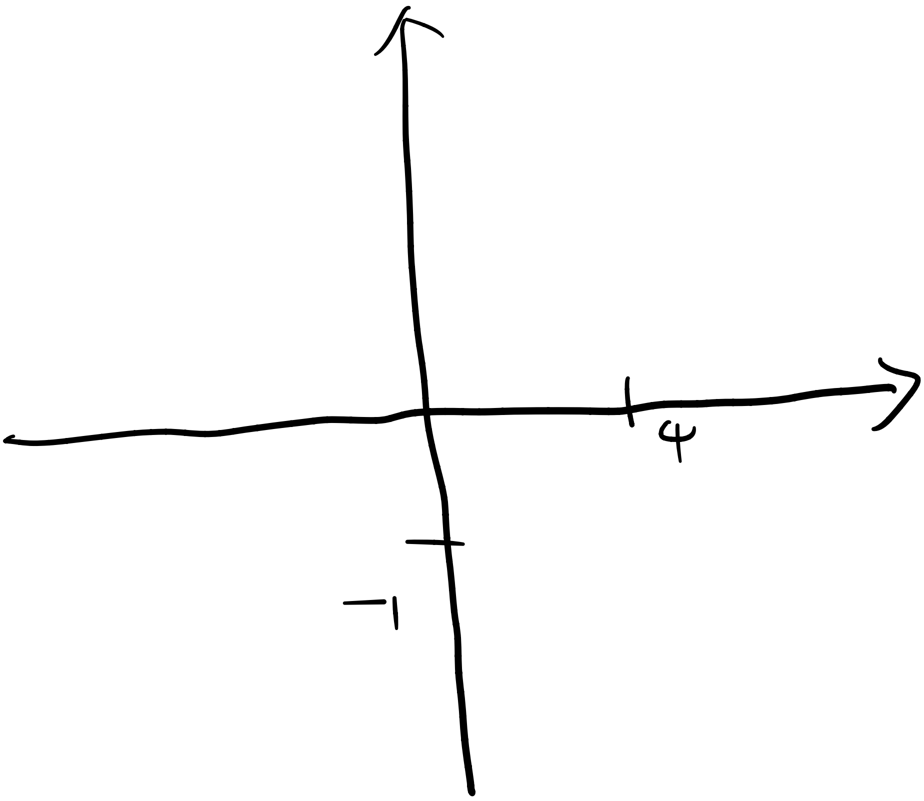
$x^2 - 8x + 18 = 0$

$x = \frac{8 \pm \sqrt{64 - 72}}{2}$

$= \frac{8 \pm \sqrt{-8}}{2}$

$= \frac{8 \pm 2\sqrt{2} i}{2}$

$= 4 \pm \sqrt{2} i$



3.7 Polynomial and Rational Inequalities

① Polynomial Inequality

② Rational Inequality

Polynomial Inequalities

7. $x^3 + 4x^2 \geq 4x + 16$

$x^3 + 4x^2 - 4x - 16 \geq 0$

Rational zero theorem: $\pm 1, \pm 2, \pm 4, \pm 8, \pm 16$

1	1	4	−4	−16
		1	5	1
	1	5	1	−15

2	1	4	−4	−16
		2	12	16
	1	6	8	0

$f(x) = (x-2)(x^2+6x+8)$

$= (x-2)(x+4)(x+2)$

$(x-2)(x+2)(x+4) \geq 0$

	-4		-2		2	
$-$		$+$		$-$		$+$

$\therefore -4 \leq x \leq -2, \; x \geq 2$

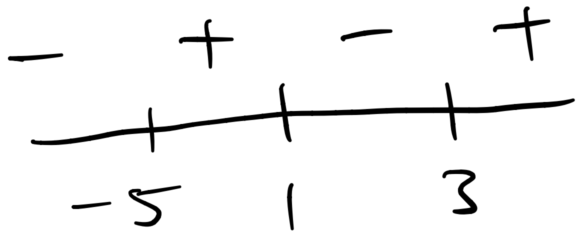
13. $x^3 + x^2 - 17x + 15 \geq 0$

RZT: $\pm 1, \pm 3, \pm 5, \pm 15$

1	1	1	−17	15
		1	2	−15
	1	2	−15	0

$$(x-1)(x^2+2x-15) \geq 0$$

$$(x-1)(x+5)(x-3) \geq 0$$



$$\therefore \; -5 \leq x \leq 1, \; x \geq 3$$

Rational Inequalities

27. $\dfrac{x-3}{2x+5} \geq 1$

$x-3 \geq 2x+5$

$-x-8 \geq 0$

$-x \geq 8$

$x \leq -8$

$\therefore -8 \leq x < \frac{5}{2}$

$\dfrac{x-3}{2x+5} - 1 \geq 0$

$\dfrac{x-3-(2x+5)}{2x+5} \geq 0$

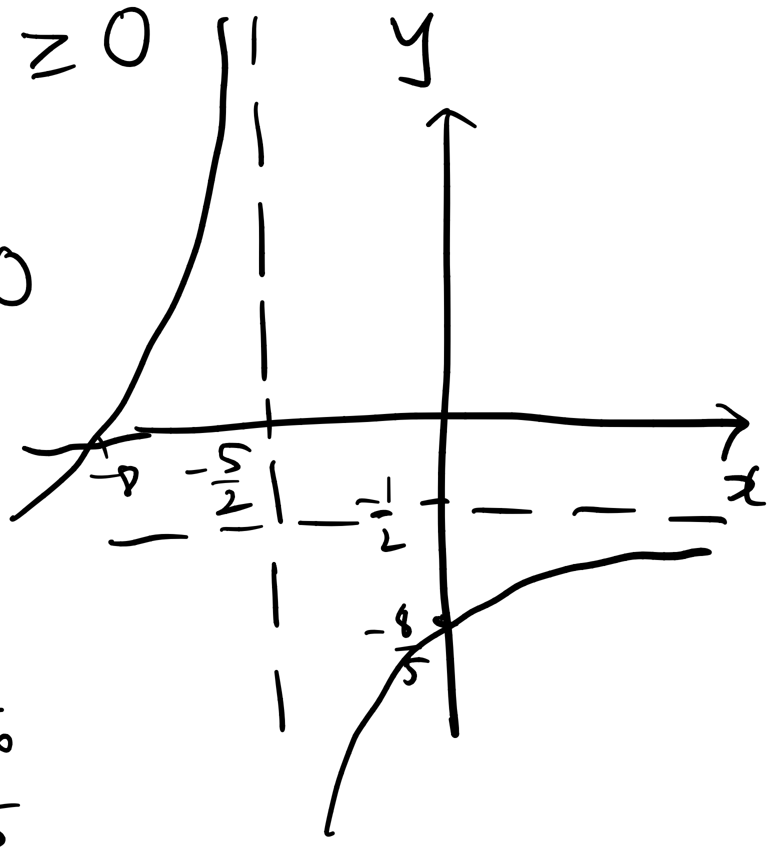
$f(-10) = \dfrac{2}{-15}$

$\dfrac{-x-8}{2x+5} \geq 0$

$f(10) = \dfrac{-18}{25}$

$=$

$\dfrac{-1-\frac{8}{x}}{2+\frac{5}{x}}$



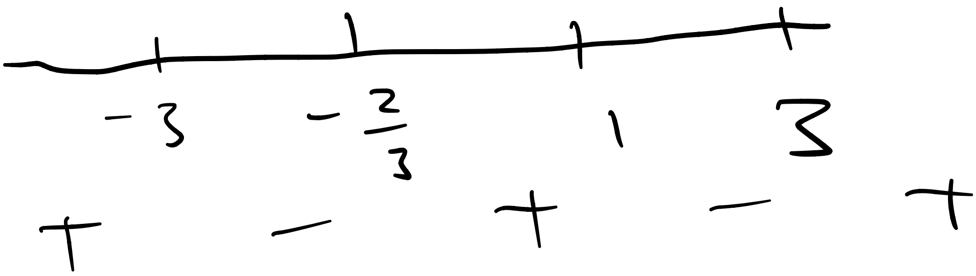
$x \to -\frac{5}{2}^{+}, f(-2) = -6$

$x \to -\frac{5}{2}^{-}, f(-3) = 5$

23. $$\frac{x^2+2x-3}{3x^2-7x-6} > 0$$

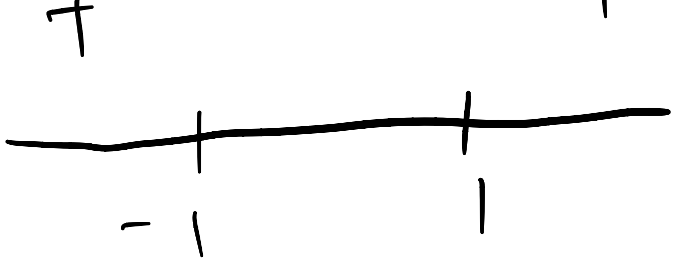
Horizontal asymptote : $y = \frac{1}{3}$

$$\frac{(x+3)(x-1)}{(3x+2)(x-3)} > 0$$



$\therefore$ $(-\infty, -3) \cup (-\frac{2}{3}, 1) \cup (3, \infty)$

24. $$\frac{x-1}{x^3+1} \geq 0$$



$x^3+1=0$

$$\begin{array}{r|rrrr} -1 & 1 & 0 & 0 & 1 \\ & & -1 & 1 & -1 \\ \hline & 1 & -1 & 1 & 0 \end{array}$$

$(x+1)(x^2-x+1)=0$

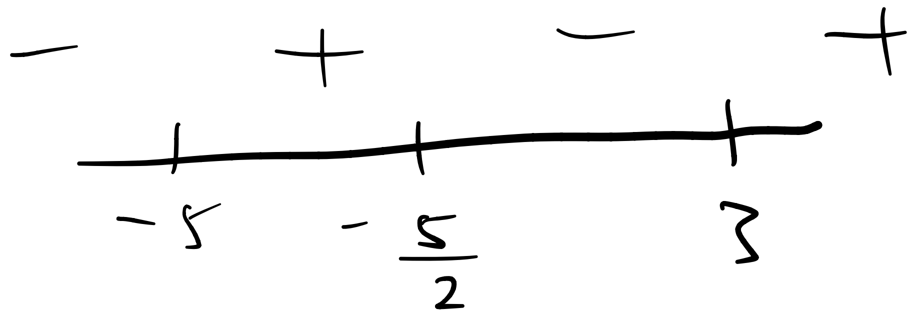
$x = \frac{1 \pm \sqrt{1-4}}{2}$

$= \frac{1 \pm \sqrt{3}i}{2}$

$\therefore (-\infty, -1) \cup [1, \infty)$

① Polynomial Inequalities

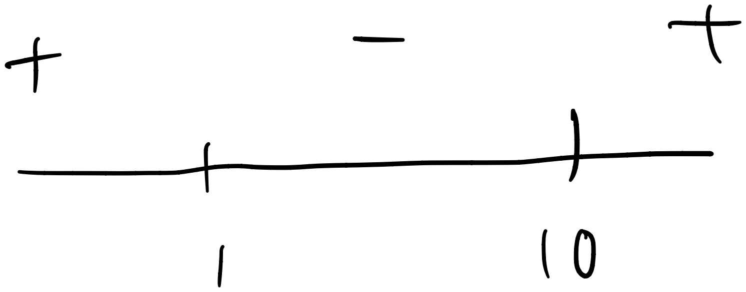
3. $(x-3)(x+5)(2x+5) < 0$



$\therefore \quad (-\infty, -5) \cup (-\frac{5}{2}, 3)$

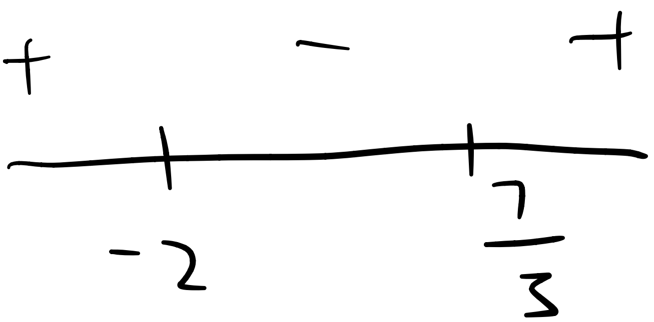
② Rational Inequalities

17. $\dfrac{x-1}{x-10} < 0$



$\therefore \; 1 < x < 10$ / $(1, 10)$

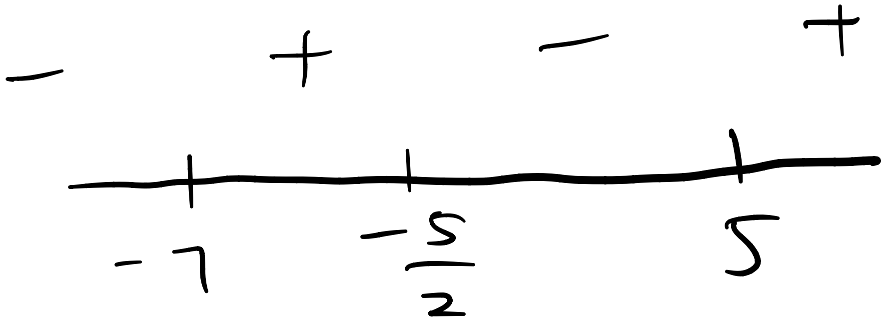
18. $\dfrac{3x-7}{x+2} \le 0$



$\therefore \; -2 < x \le \frac{7}{3}$ [illegible] $\left(-2, \frac{7}{3}\right]$

19. $$\frac{2x+5}{x^2+2x-35} \geq 0$$

$$\frac{2x+5}{(x+7)(x-5)} \geq 0$$

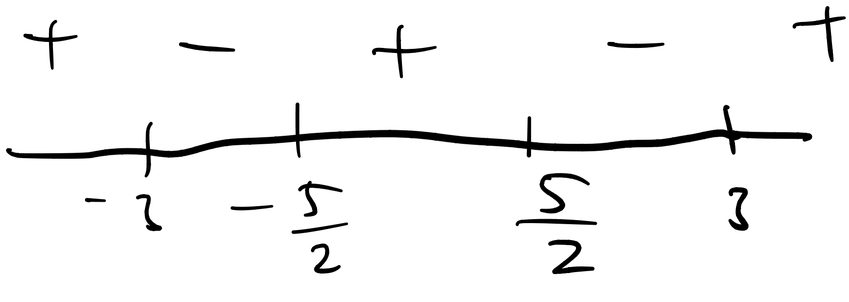


$$\therefore \left(-7, -\frac{5}{2}\right] \cup (5, \infty)$$

20. $$\frac{4x^2-25}{x^2-9} \leq 0$$

$$\therefore \left(-3, -\frac{5}{2}\right] \cup \left[\frac{5}{2}, 3\right)$$

$$\frac{(2x+5)(2x-5)}{(x+3)(x-3)} \leq 0$$



Domain of a Function 41 – 44

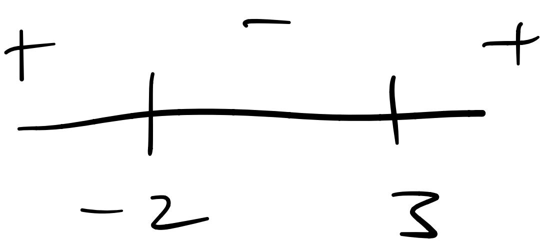
41. $f(x) = \sqrt{6 + x - x^2}$

square root has to be non-negative,

$$6 + x - x^2 \geq 0$$

$$x^2 - x - 6 \leq 0$$

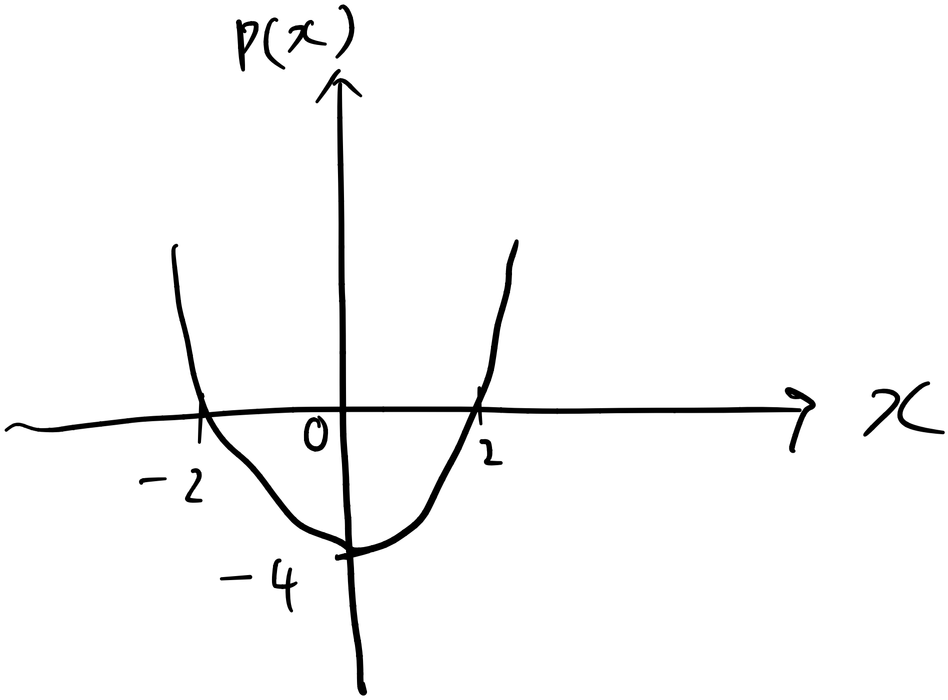
$$(x-3)(x+2) \leq 0$$



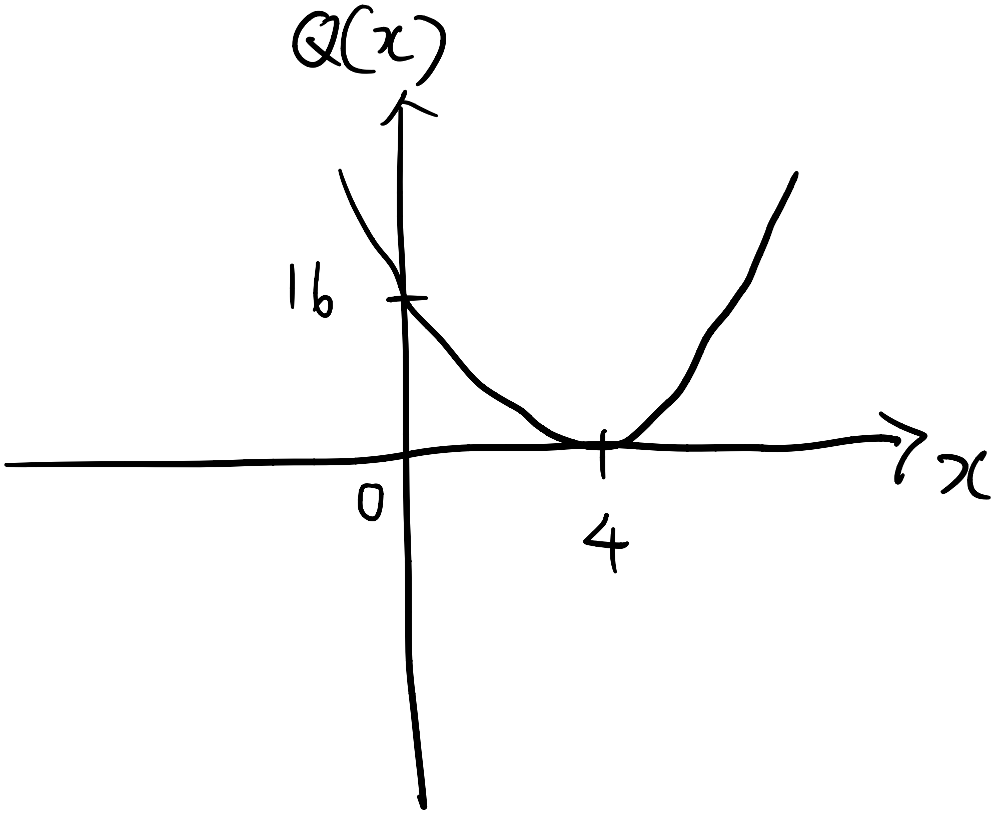
$\therefore$ Domain : $[-2, 3]$

3.2 Polynomial Functions and Their Graphs

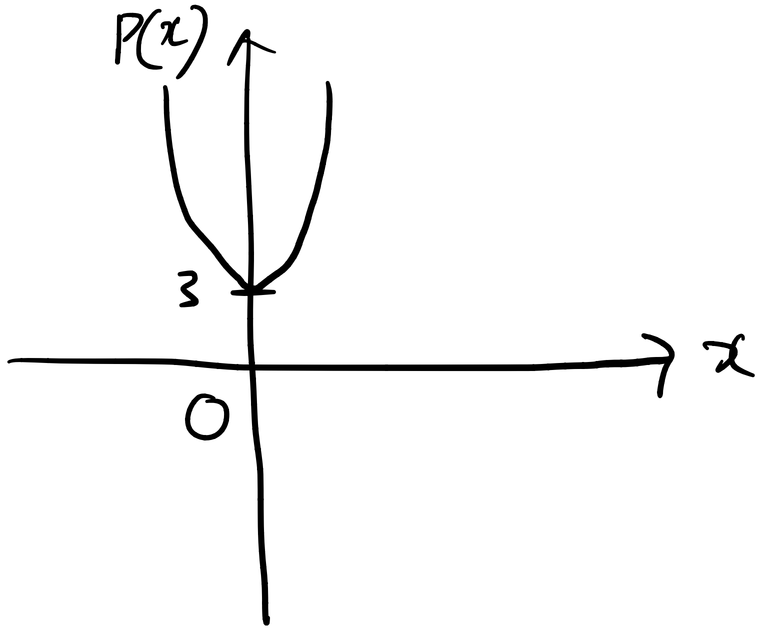
5. (a) $P(x) = x^2 - 4$



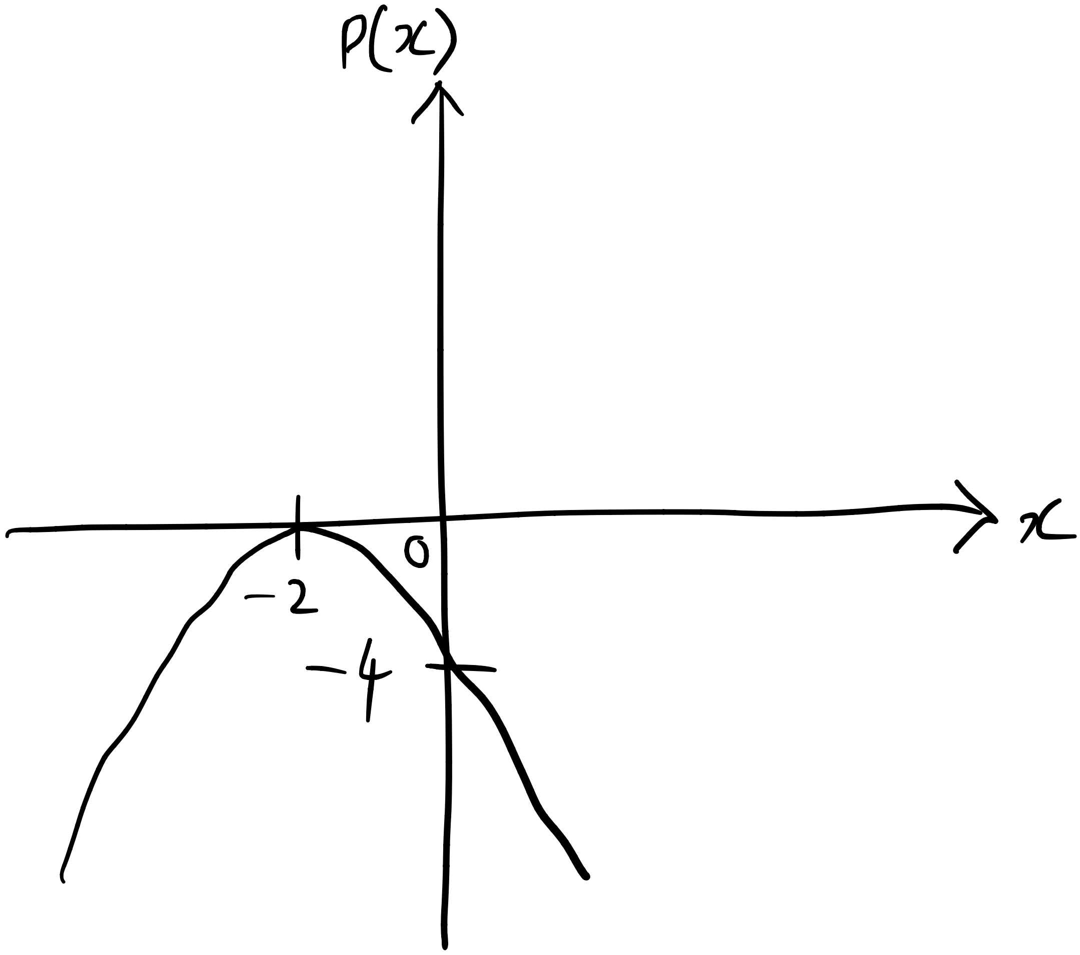
(b) $Q(x) = (x-4)^2$



(c) $P(x) = 2x^2 + 3$



(d) $P(x) = -(x+2)^2$



11. $R(x) = -x^5 + 5x^3 - 4x$

(a) $x \to -\infty, \; y \to \infty$

$x \to \infty, \; y \to -\infty$

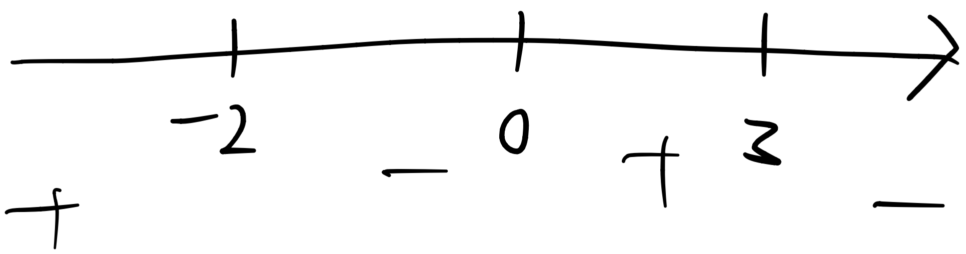
(b) IV

45. $P(x) = 3x^3 - x^2 + 5x + 1 \; ; \; Q(x) = 3x^3$

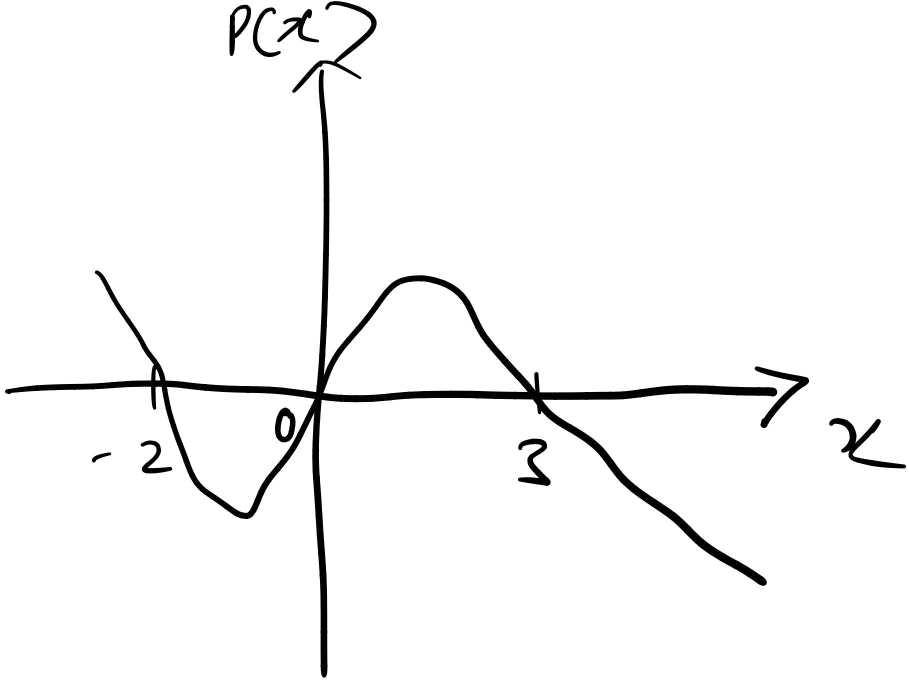
$$P(x) = 3x^3\left(1 - \frac{1}{3x} + \frac{5}{3x^2} + \frac{1}{3x^3}\right)$$

17. $P(x) = -x(x-3)(x+2)$

Zeros: $-2, 0, 3$



x	$P(x)$
-3	18
-1	-4
1	6
4	-24



65. $y = x^3 - x^2 - x$

$= x(x^2 - x - 1)$

67. $y = x^4 - 5x^2 + 4$

3.3 Dividing Polynomials

3. $P(x) = 2x^2 - 5x - 7$, $D(x) = x - 2$

$$
\begin{array}{r}
2x - 1 \\
x - 2 \overline{\smash{)}\, 2x^2 - 5x - 7} \\
2x^2 - 4x \\
\hline
-x - 7 \\
-x + 2 \\
\hline
-9
\end{array}
$$

$$\frac{P(x)}{D(x)} = 2x - 1 - \frac{9}{x-2}$$

19. $\dfrac{x^3 + 2x + 1}{x^2 - x + 3}$

$$
\begin{array}{r}
x + 1 \\
x^2 - x + 3 \overline{\smash{)}\, x^3 + 0x^2 + 2x + 1} \\
x^3 - x^2 + 3x \\
\hline
x^2 - x + 1 \\
x^2 - x + 3 \\
\hline
-2
\end{array}
$$

$$P(x) = D(x) \cdot Q(x) + R(x)$$
$$= (x^2 - x + 3)(x + 1) - 2$$

31. $\dfrac{x^3-8x+2}{x+3}$

-3	1	0	-8	2
		-3	9	-3
	1	-3	1	-1

$$\frac{x^3-8x+2}{x+3} = (x+3)(x^2-3x+1)-1$$

39. $p(x) = 4x^2+12x+5, \quad c=-1$

-1	4	12	5
		-4	-8
	4	8	-3

$$4x^2+12x+5 = (x+1)(4x+8)-3$$

$$p(-1) = 4(-1)^2+12(-1)+5$$
$$= 4-12+5$$
$$= -3$$

53. $P(x) = x^3 - 3x^2 + 3x - 1$, $c = 1$

$$P(1) = 1^3 - 3(1)^2 + 3(1) - 1$$
$$= 1 - 3 + 3 - 1$$
$$= 0 \text{ (shown)}$$

57. $P(x) = x^3 + 2x^2 - 9x - 18$, $c = -2$

$$P(-2) = (-2)^3 + 2(-2)^2 - 9(-2) - 18$$
$$= -8 + 8 + 18 - 18$$
$$= 0$$

-2	1	2	-9	-18
		-2	0	18
	1	0	-9	0

$$P(x) = (x^2 - 9)(x - (-2))$$
$$= (x+2)(x+3)(x-3)$$

63. Degree 3, zeros: $-1, 1, 3$

$$P(x) = (x+1)(x-1)(x-3)$$
$$= (x^2 - 1)(x-3)$$
$$= x^3 - x - 3x^2 + 3 = x^3 - 3x^2 - x + 3$$

67. Degree 4, zeros $-2, 0, 1, 3$ coef. of x^3 is 4

$$P(x) = (x+2)(x)(x-1)(x-3)$$

$$= x(x-1)(x-3)(x+2)$$

$$= (x^2-x)(x^2-x-6)$$

$$= x^4-x^3-6x^2-x^3+x^2+6x$$

$$= x^4-2x^3-5x^2+6x$$

$$P(x) = -2x^4+4x^3+10x^2-12x$$

3.4 Real Zeros of Polynomials

15. $P(x) = x^3 + 2x^2 - 13x + 10$

$P(2) = 8 + 8 - 26 + 10 = 0$

$P(5) = 125 + 50 - 65 + 10 = 170$

$P(10) = 1000 + 200 - 130 + 10 \neq 0$

$P(1) = 1 + 2 - 13 + 10 = 0$

$P(-1) = -1 + 2 + 13 + 10 \neq 0$

$P(-2) = -8 + 8 + 26 + 10 \neq 0$

$P(-5) = -125 + 50 + 65 + 10 = 0$

$P(-10) = -1000 + 200 + 130 + 10 \neq 0$

$P(x) = (x-1)(x-2)(x+5)$

29. $p(x) = 4x^4 - 37x^2 + 9$

$p\left(\frac{3}{2}\right) \neq 0$

$p(3) = 324 - 333 + 9 = 0$

-3	4	0	-37	0	9
		-12	36	3	-9
	4	-12	-1	3	0

$p(x) = (x-3)(4x^3 - 12x^2 - x + 3)$

3	4	-12	-1	3
		12	0	-3
	4	0	-1	0

$p(x) = (x-3)(x+3)(4x^2 - 1)$

$= (x-3)(x+3)(2x+1)(2x-1)$

45. $p(x) = 3x^3 + 5x^2 - 2x - 4$

$p(4) = 3(64) + 5(16) - 2(4) - 4$

$= 192 + 80 - 8 - 4$

$\neq 0$

$p(2) = 3(8) + 5(4) - 2(2) - 4$

$= 24 + 20 - 4 - 4 \neq 0$

$p(-2) = 3(-8) + 5(4) - 2(-2) - 4$

$= -24 + 20 + 4 - 4$

$\neq 0$

$p(-4) = 3(-64) + 5(16) + 8 - 4 \neq 0$

$p(1) = 3 + 5 - 2 - 4 \neq 0$

$p(-1) = -3 + 5 + 2 - 4 = 0$

$p\left(\frac{2}{3}\right) = 3\left(\frac{8}{27}\right) + 5\left(\frac{4}{9}\right) - 2\left(\frac{2}{3}\right) - 4$

$= \frac{8}{9} + \frac{20}{9} - \frac{4}{3} - \frac{36}{9}$

$= \frac{28}{9} - \frac{12}{9} - \frac{36}{9} \neq 0$

$p\left(-\frac{2}{3}\right) = 3\left(-\frac{8}{27}\right) + 5\left(\frac{4}{9}\right) - 2\left(-\frac{2}{3}\right) - 4$

$= -\frac{8}{9} + \frac{20}{9} + \frac{4}{3} - \frac{36}{9}$

$= \frac{12}{9} + \frac{12}{9} - \frac{36}{9}$

$$P\left(\frac{4}{3}\right) = 3\left(\frac{64}{27}\right) + 5\left(\frac{16}{9}\right) - 2\left(\frac{4}{3}\right) - 4$$

$$= \frac{64}{9} + \frac{80}{9} - \frac{24}{9} - \frac{36}{9}$$

$$\neq 0$$

$$P\left(-\frac{4}{3}\right) = 3\left(-\frac{64}{27}\right) + 5\left(\frac{16}{9}\right) - 2\left(-\frac{4}{3}\right) - 4$$

$$= -\frac{64}{9} + \frac{80}{9} + \frac{24}{9} - \frac{36}{9}$$

$$= \frac{16}{9} + \frac{24}{9} - \frac{36}{9} \neq 0$$

$$P\left(\frac{1}{3}\right) = 3\left(\frac{1}{27}\right) + 5\left(\frac{1}{9}\right) - 2\left(\frac{1}{3}\right) - 4$$

$$= \frac{3}{27} + \frac{15}{27} - \frac{18}{27} -$$

$$\neq 0$$

$$P\left(-\frac{1}{3}\right) = -\frac{3}{27} + \frac{15}{27} -$$

$$\neq 0$$

$$\begin{array}{r|rrrr} -1 & 3 & 5 & -2 & -4 \\ & & -3 & -2 & 4 \\ \hline & 3 & 2 & -4 & 0 \end{array}$$

$$P(x) = (x+1)(3x^2+2x-4)$$

$$x = \frac{-2 \pm \sqrt{4+48}}{6} = \frac{-2 \pm \sqrt{52}}{6} = \frac{-2 \pm 2\sqrt{13}}{6} = \frac{-1 \pm \sqrt{13}}{3}$$

$$\therefore -1, \frac{-1 \pm \sqrt{13}}{3}$$

63. $p(x) = x^3 - x^2 - x - 3$

1 positive real zero

$$p(-x) = (-x)^3 - (-x)^2 - (-x) - 3$$
$$= -x^3 - x^2 + x - 3$$

either 2 or 0 negative zeros

$\therefore$ 3 or 1 real zeros

69. $P(x) = 2x^3 + 5x^2 + x - 2$; $a = -3, b = 1$

$$\begin{array}{r|rrrr} -3 & 2 & 5 & 1 & -2 \\ & & -6 & 3 & -12 \\ \hline & 2 & -1 & 4 & -14 \end{array}$$

$\therefore$ alternate signs, lower bound

$$\begin{array}{r|rrrr} 1 & 2 & 5 & 1 & -2 \\ & & 2 & 7 & 8 \\ \hline & 2 & 7 & 8 & 6 \end{array}$$

$\therefore$ no negative entry, upper bound

81. $P(x) = 2x^4 + 3x^3 - 4x^2 - 3x + 2$

2	2	3	−4	−3	2
		4	14	20	34
	2	7	10	17	36

1	2	3	−4	−3	2
		2	5	1	−2
	2	5	1	−2	0

$P(x) = (2x^3 + 5x^2 + x - 2)(x+1)$

−1	2	5	1	−2
		−2	−3	2
	2	3	−2	0

$P(x) = (x+1)(x-1)(2x^2 + 3x - 2)$

$= (x+1)(x-1)(2x-1)(x+2)$

Quadratic Formula